රාජකීය විද්‍යාලය - කොළඹ 07
Royal College - Colombo 07

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2025
General Certificate of Education (Adv. Level) Examination, 2025

සංයුක්ත ගණිතය I
Combined Mathematics I

10 E I

Part B

- Answer only **four** Questions.

09. (a) Let $F(x) \equiv (a-2b+c)x^2 + (a+b+c)x - 2a + b - 2c$ such that $(a,b,c) \in \mathbb{R}$. If it is given that a, b, c **do not** lie as consecutive terms of an arithmetic series, show that the equation $F(x)=0$ has a couple of roots. Without solving the equation, show that for all $(a, b, c) \in \mathbb{R}$ that 1 is a root of $F(x)=0$ Further writhing down the discriminant of $F(x)=0$ in terms of a, b, c deduce that the above couple of roots is real. Now, let $(a, b, c) \equiv (k, 1, k)$ write down expressions for the sum and product of roots in terms of k. Give the range of values for k in interval notation such that one of the roots of $F(x)=0$ is greater than 1.

(b) Let f is a polynormial function of degree 3. It is given that the remainder is 7 when the general image $f(x)$ of f is separately divided by $(x-1)$, $(x-2)$ and $(x-3)$. Further if it is given that (-5) is the image of zero on the function f, find $f(x)$ and write down it in standard notation. Further, deduce the set of solutions of x such that. $f(x-1) - 22x + 27 \leq 0$

10. (a) (i) Sketch the graph of $y = x|x-4|$. Using the graph find the set of all real values of x satisfying the inequality $x|x-4| - 2 \geq 0$. **Hence or otherwise deduce** the set of all real values of x satisfying the inequality $x|x-8| - 8 \geq 0$.

(ii) Determine the set of all the real values of x satisfying the inequality $\left|\dfrac{4x}{x+2}\right| \geq 4 - x$ using either the graphical method or the algebraic method.

(b) Using laws of indices, show that $\log_a b = \dfrac{\log_c b}{\log_c a}$ then deduce $\log_{(a^q)} x^p = \dfrac{p}{q} \log_a x$

Here $a, b, x > 0$, $a, c, \neq 1$, and $p, q \in \mathbb{Q}$ where $q \neq 0$

Hence, Show that

$$\log_{a^{(2r-1)!}} x^{(2r)!} + \log_{a^{(4r-1)!}} x^{(4r)!} + \log_{a^{(6r-1)!}} x^{(6r)!} + \cdots + \log_{a^{(100r-1)!}} x^{(100r)!} = \log_a x^{2550r}$$

$r \in \mathbb{Z}_0^+$, $n \in \mathbb{Z}_0^+$ that $n! = 1 \times 2 \times 3 \times \cdots \times n$

11. (a) (i) Prove that for positive integers n, $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ Deduce the result for negative integers as well. Hence show that the above theorem is true for any rational number n.

(taking $n = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$)

(ii) Evaluate $n(n \in \mathbb{Q})$ such that $\lim_{x \to 1} \frac{\sqrt[4]{3x+13} - 2}{x^3 - 1} = \frac{1}{4^n}$

(iii) Find $k(k \in \mathbb{R})$ such that $\lim_{x \to \frac{\pi}{4}} \left(\frac{\sec^2 x - \sqrt{32}}{\sin x - \cos x} \right) \left(\frac{64x^3 - \pi^3}{1 - \sqrt{2}\cos x} \right) = k\pi^2$

(b) Decompose $\frac{2x^4 + 5}{(x^2 - 1)(x^2 + 2)}$ (for $x \neq \pm 1$) in to partial fractions

Hence deduce the partial fractions of

$\frac{2x^4 + 80}{(x^2 - 4)(x^2 + 8)}$ for $x \neq \pm 2$

12. (a) Let f is a polynomial function of degree 2. It is given that 4 is **the least image** on function f and that $y = f(x)$ is symmetrical about the line $x = 1$. Further find $f(x)$ such that the point $A \equiv (2, 5)$ lies on $y = f(x)$

Writing down the domain and co domain of function f, sketch graph of $y = f(x)$ on a Cartesian plane.

Now, let $g(x) = \sqrt{f(x) - 4}$ for $x \leq 1$. Write down $g(x)$ in terms of x and express the inverse function g^{-1} of g in terms of x. Writing down the domain and co domain of g^{-1}, sketch the graph of g^{-1} on a cartesian plane.

Moreover state **graphically** the set of real images of $y = \left| g^{-1}\sqrt{x} \right|$ for $x \geq 0$

(b) Sketch $F(x) = \left| x^2 - 4x + 1 \right|$. Write down the coordinates of points A and B where the graph intersects with the x axis. If C is the **maximum** point of the curved section between A and B, write down the coordinates of C.

Further,

(i) Show that the lines AC and BC are **not** perpendicular to each other

(ii) Determine the area of the circle S, in terms of π in which AC is the radius and C is the center.

(iii) Two tangents are drawn to the circle S from the point $D \equiv \left(2+2\sqrt{3},\ 1\right)$. If these two tangents intersect the circle S at points X and Y, determine the area of the quadrilateral.

(iv) If Q is the point where the line DC intersects with the circle S, Show that $Q \equiv \left(5,\ 3-\sqrt{3}\right)$

13. (a) If $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\theta$, show that $\frac{x^2}{a^2}-\frac{2xy\cos\theta}{ab}+\frac{y^2}{b^2}=\sin^2\theta$.

(b) Express $f(\theta)=\cos^6\theta+\sin^6\theta$ in the form $f(\theta)\equiv a+b\cos(4\theta+\alpha)$. Here $a(>0)$, b and α are constants to be determined. Deduce the maximum and minimum values $f(\theta)$ can take. Sketch the graph of $y=f(\theta)$ for $0\leq\theta\leq\pi$. Hence detemine the solutions satisfied by $\cos 4\theta+1=0$ in $0\leq\theta\leq\pi$.

(c) State and prove the cosine rule for a triangle ABC in standard notation.

$a=4$ and $b=3$ in the triangle ABC. If medians drawn from points A and B are perpendicular to each other, show that $\cos C=\frac{5}{6}$.

රාජකීය විද්‍යාලය - කොළඹ 07
Royal College - Colombo 07

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2025
General Certificate of Education (Adv. Level) Examination, 2025

සංයුක්ත ගණිතය II
Combined Mathematics II

10 E II

පැය දෙකයි මිනිත්තු තිහයි.
Two hours and thirty minutes.

Part B

- Answer only four Questions.

09. (a) A ball P is projected vertically upward with velocity u from a point A at a vertical height of $2h$ above the ground level. In the same instant,another equivalent ball Q at point B, which is h height above the level of point A, is projected vertically downward at velocity ku $(0<k<1)$. Sketch the velocity time graph of both balls is the same diagram, until P reaches its maximum height Hence,

i. If P and Q collide, show that the time taken for collision is $\dfrac{h}{u(1+k)}$

ii. Find the velocity of P at the moment of collision

iii. Show that the vertical height to the point of collision from A is $\dfrac{h\left(2u^2k-gh+2u^2\right)}{2u^2\left(1+k\right)^2}$

iv. If both balls are in motion without collision, for Q to hit the ground before P reaches its maximum height, show that $k>\dfrac{6gh-u^2}{2u^2}$

(b) Two soldiers are on guard duty at the top A and base O of an observation tower OA of height h. Both are holding two guns of same type. In a certain moment a drone D is observed held at height $b(b>h)$ from ground and at a horizontal distance a from O. Both soldiers aim the drone and shoot. If bullets released from points A and O are A' and O' and their projected velocity is u and their angles of projections are α and θ to horizontal $(\theta>\alpha)$ respectively

i. Develop the cartesion equation of the bullet O' referred to a horizontal and vertical axes taking through origin O.

ii. By suitable substitution of x, y and θ deduce the cartesian equation of A'

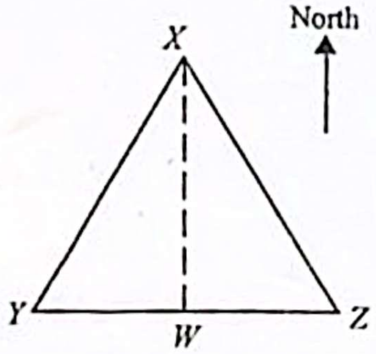
iii. Show thatthe horizontal range of O' is given by $\dfrac{2u^2\tan\theta}{g\sec^2\theta}$. Hence show that if $u<\sqrt{ag}$, O' bullet will never reach D.

iv. Show that, $ga^2\tan^2\theta-2u^2a\tan\theta+2u^2b+a^2g=0$ should be, for O' bullet to hit D.

v. If given that $2b = a$, show that, $u^4 - agu^2 - a^2g^2 > 0$ should be, for there to be two angles to shoot from O to hit D.

vi. Show that for bullets O' and A' to collide D at the same time, bullet A' should be released $\frac{a}{u}\left\{\frac{\cos\alpha - \cos\theta}{\cos\alpha\cos\theta}\right\}$ time after releasing O'.

10. (a) Three flags X, Y and Z are planted in the ground to form an equilateral triangle as demonstrated in below diagram. $XY = a$. In a race, at initial moment a cyclist starts from point X and goes to point Y, then from Y to Z and then move from Z to X. The velocity of the cyclist in still air is v. The mid point of YZ is W. A wind blows horizontally along XW at a constant velocity u $u(< v)$.



If time taken from X to Y is t_1

If time taken from Y to Z is t_2

If time taken from Z to X is t_3

i. Show that $t_1 < t_3$

ii. Show that the time taken for the cyclist to return back to X is $\dfrac{a\left(\sqrt{4v^2 - u^2} + \sqrt{v^2 - u^2}\right)}{v^2 - u^2}$

(b) At the moment the above cyclist reach the point Y, a dog at point S, 12 units south of point Y, moves with a constant velocity $\dfrac{\sqrt{v^2 - u^2}}{2}$ in a road with a longitude 030°.

i. Show that the dog does not get hit by the cycle

ii. Show that the shortest distance between the dog and the cycle is $6\sqrt{3}$

iii. Find the time from the start of the race to cause the shortest distance between the dog and the cycle.

11. Define the Dot product for two non-zero vectors.

Take $ABCD$ as a square such that AB parallel to DC. If the position vectors of points A, B and D with respect to origine O are given as $\mathbf{a} = -(\mathbf{i} + \mathbf{j})$, $\mathbf{b} = \mathbf{i} + 4\mathbf{j}$ and $\mathbf{d} = 4\mathbf{i} - 2\mathbf{j}$, find the value of α, such that the position vector of $\mathbf{c} = \alpha(\mathbf{i} + \mathbf{j})$.

Now, take the diagonals AC and BD intersect at E. By considering the triangle OAE or otherwise show that $\overline{OE}$ could be described as $\overline{OE} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{c}$. Further, by considering the triangle OBE or otherwise show that of could also be described as $\overline{OE} = \mu\mathbf{b} + (1 - \mu)\mathbf{d}$ and evaluate λ and μ.

Hence, find the ratio that point E intersect the diagonals.

Evaluate $A\hat{D}B$ in terms of $\overline{AD} \cdot \overline{BD}$

Also, evaluate $A\hat{C}B$ and hence deduce that $ABCD$ is not a cyclic quadrilateral.

12. (a) Consider the general force system with force $F_r = (X_r, Y_r)$ acting at the point $P_r = (x_r, y_r)$ Here, $r = 1, 2, 3, ..., n$

Show that this system of coplanar forces can be reduced to a single force R acting at origin O together with a couple G and state the direction and magnitude of the single force R.
Further show that, the single force R acting through origin O and couple G can be reduced to a single force R' acting at point $P \equiv (x', y')$ in the plane and to a couple G'. Here $R' = R$ and $G' = G - x'Y + y'X$. Here X and Y are the algebraicsums of horizontal and vertical components of the system of coplanar forces respectively.
Show that this single force R' and couple G' is reduced to a single force with equation of line of action $G - xY - yX = 0$

(b) With referred to rectangular axes Ox and Oy the coordinates of points A, B and C are $(0, 0)$, $(2a, 0)$ and $(a, \sqrt{3}a)$ respectively. A force system is applied on this plane and the anticlockwise sense moments about the points A, B and C of that force system are $\frac{G}{2}$, $2G$ and $3G$ respectively. By using the result obtained for G' or otherwise find the horizontal and vertical components of the system and show that a resultant force of magnitude $\frac{\sqrt{19}}{2}Ga$ is acting with inclination $\tan^{-1}\left(3\sqrt{3}\ \frac{G}{7}\right)$ from east to south.

Also use the previously obtained result to show that the equation of line of action is $3\sqrt{3}x + 7y + 2\sqrt{3}a = 0$

13. (a) A straight uniform rod with weight W is kept completely inside a fixed rough hemispherical bowl with rim horizontal. The end points of the rod subtends an angle 2θ with the centre of the bowl. Using the angle of friction λ, show that the rod cannot be horizontal, when it is in limiting equilibrium. (hint : take the angle that rod makes with upward vertical as α)

(b) Further, when $\theta = \frac{\pi}{4}$ and angle of friction is $\frac{\pi}{8}$, for the rod to kept horizontally in limiting equilibrium, show that the ratio by which length of the unequal rod be divided by its centre of gravity is $k+1 : k-1$. Here $(k \in \mathbb{R})$ is a constant to be determined.
Also find the normal reaction generated at the both touching ends of the rod and hence show tha the ratio between the resultant forces (total) is $k-1 : k+1$

• • •